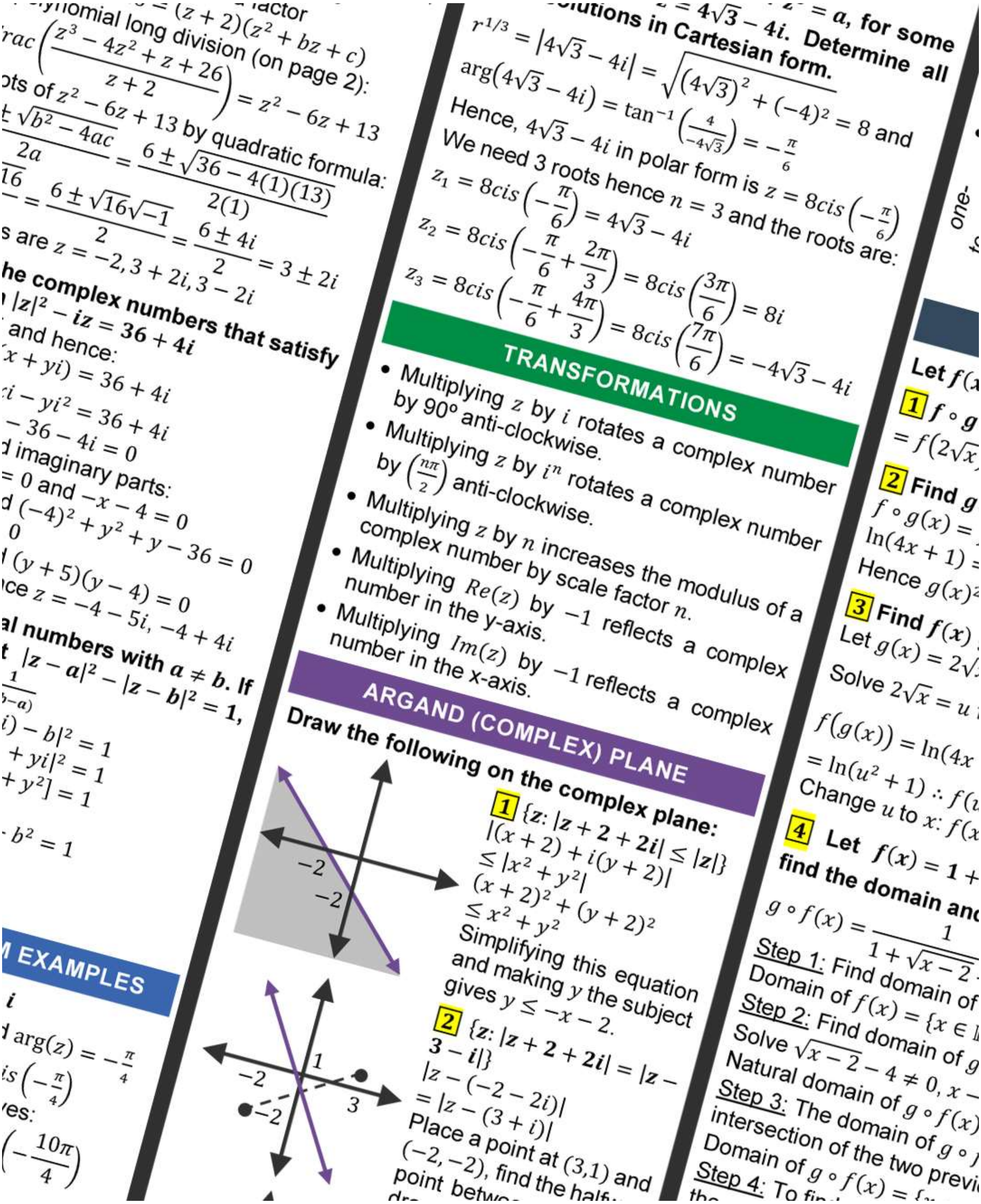

ATAR Mathematics Specialist Units 3 & 4

Exam Notes for WA Year 12 Students

Created by Anthony Bochrinis

Version 1.4 (22/09/17)

Year 12 ATAR Mathematics Specialist Units 3 & 4 Exam Notes

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About the Creator – Anthony Bochrinis

I graduated from high school in 2012, completed a Bachelor of Actuarial Science in 2015 and am currently completing my Graduate Diploma in Secondary Education with the goal of becoming a full-time high school teacher next year!

My original exam notes (created in 2013) were inspired by Severus Snape's copy of Advanced Potion Making in Harry Potter and the Half-Blood Prince; a textbook filled with annotations containing all of the pro tips and secrets to help gain a clearer understanding.

I hope that my exam notes help to sharpen your knowledge and I wish you all the best in your exams!

Using these Exam Notes

These exam notes are designed to be a complement to your studies throughout the year. As such, I recommend using these exam notes during class, during tests, whilst studying at home or in the library and even in the calculator-assumed section of your mock and WACE exams.

These exam notes contain theory, diagrams, formulae and worked examples based off the official SCSA syllabus to give you a full revision of the entire course in just 4 pages. For more detailed information about our most frequently asked questions about the use of these exam notes, please visit my website or email me.

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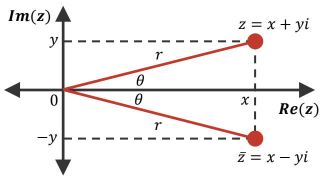
... without further ado, I present my exam notes!

COMPLEX NUMBERS

IMAGINARY NUMBERS

$i^{-4}=1$	$i^0=1$	$i^4=1$
$i^{-3}=\sqrt{-1}$	$i^1=\sqrt{-1}$	$i^5=\sqrt{-1}$
$i^{-2}=-1$	$i^2=-1$	$i^6=-1$
$i^{-1}=-i$	$i^3=-i$	$i^7=-i$

COMPLEX NUMBER NOTATION



- Im: imaginary axis (vertical axis)
- Re: real axis (horizontal axis)
- z: complex number ($z=x+yi$)
- $\bar{z}$: conjugate of a complex number ($\bar{z}=x-yi$) and is reflected in the real axis
- x: real components (horizontal axis)
- y: imaginary component (vertical axis)
- r: modulus (length) of a complex number and can also be represented by $|z|$
- θ: argument (angle that the complex number makes with the real axis) of a complex number and can also be represented by $\arg(z)$

RECTANGULAR (CARTESIAN) FORM

- $z=x+yi$ where:
  - x: is the real component
  - y: is the imaginary component

Convert Polar to Rectangular (Cartesian):
- $x=r\times\cos(\theta)$ and $y=r\times\sin(\theta)$

Distance between two points A and B:
- $\overline{AB}=\sqrt{(x_B^2-x_A^2)^2+(y_B^2-y_A^2)^2}$

POLAR FORM

- $z=r cis(\theta)$ where:
  - r: is the modulus
  - θ: is the argument
  - $cis(\theta)$: is short for $\cos(\theta)+i sin(\theta)$

Convert Rectangular (Cartesian) to Polar:
- $r=|z|=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

Distance between two points A and B:
- $\overline{AB}=\sqrt{r_A^2+r_B^2-2r_Ar_B\cos(\theta_A-\theta_B)}$

COMPLEX NUMBER RULES

Rules for Complex Conjugates

$\overline{z_1\pm z_2}=\bar{z_1}\pm\bar{z_2}$	$\overline{z_1\times z_2}=\bar{z_1}\times\bar{z_2}$
$\bar{z}=x-yi=r cis(-\theta)$	
$z+\bar{z}=2Re(z)=2x=2r\cos\theta$	
$z-\bar{z}=2iIm(z)=2yi=2r(i\sin\theta)$	
$z\times\bar{z}=x^2+y^2=\lvert z\rvert^2=r^2$	
$\frac{z}{\bar{z}}=\left(\frac{x^2-y^2}{x^2+y^2}\right)+i\left(\frac{2xy}{x^2+y^2}\right)=cis(2\theta)$	

Rules for Arguments

$\arg(z\times w)=\arg(z)+\arg(w)$
$\arg(z\div w)=\arg(z)-\arg(w)$

Rules for Moduli

$\lvert z\times w\rvert=\lvert z\rvert\times\lvert w\rvert$	$\left\lvert\frac{z}{w}\right\rvert=\frac{\lvert z\rvert}{\lvert w\rvert}$

More Complex Number Rules

$z^{-1}=\frac{1}{z}=\frac{1}{x+yi}\times\frac{x-yi}{x-yi}=\frac{x-yi}{x^2+y^2}=\frac{\bar{z}}{\lvert z\rvert}$
$\frac{z}{w}=\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\times\frac{c-di}{c-di}=\frac{z\times\bar{w}}{\lvert w\rvert^2}$

DE MOIVRE'S THEOREM

- $(r cis\ \theta)^n=r^n\cos(n\theta)+r^n isin(n\theta)$
- $z^n=|z|^n cis(n\theta)$
- $z^{\frac{1}{n}}=|z|^{1/n}\left[cis\left(\frac{\theta+2\pi k}{n}\right)\right]$ for an integer k
- Find the complex n^{th} roots of a non-zero complex number z:
  - Step 1: Write z in polar form: $z=r(cis\theta)$
  - Step 2: z will have n different n^{th} roots (i.e. 3 cube roots, 4 fourth roots etc.)
  - Step 3: All these roots will have the same modulus $|z|^{1/n}=r^{1/n}$
  - Step 4: Roots have different arguments: $\frac{\theta}{n},\frac{\theta+(1\times2\pi)}{n},\frac{\theta+(2\times2\pi)}{n},\ldots,\frac{\theta+((n-1)\times2\pi)}{n}$
  - Step 5: The complex n^{th} roots of z are given in polar form by:
    - $z_1=r^{1/n}cis\left(\frac{\theta}{n}\right)$
    - $z_2=r^{1/n}cis\left(\frac{\theta+(1\times2\pi)}{n}\right)$
    - $z_3=r^{1/n}cis\left(\frac{\theta+(2\times2\pi)}{n}\right)$ and so on…
    - $z_n=r^{1/n}cis\left(\frac{\theta+((n-1)\times2\pi)}{n}\right)$

COMPLEX NUMBER EXAMPLES

1 Express $\frac{4+3i}{2-i}$ in cartesian form.

$\frac{4+3i}{2-i}=\frac{4+3i}{2-i}\times\frac{2+i}{2+i}=\frac{(4+3i)(2+i)}{(2-i)(2+i)}$

$=\frac{8+4i+6i+3i^2}{4-i^2}=\frac{5+10i}{5}=1+2i$

2 Express $(-\sqrt{3}+i)(4+4i)$ in polar form.

Converting $(-\sqrt{3}+i)$ to polar form:

$r=|z|=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=\sqrt{4}=2$

$\theta=\arg(z)=tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)=-\frac{\pi}{6}$ but as z is in the second quadrant, $\arg(z)=-\frac{\pi}{6}+\pi=\frac{5\pi}{6}$

Converting $(4+4i)$ to polar form:

$r=|z|=\sqrt{4^2+4^2}=\sqrt{32}=\sqrt{16}\sqrt{2}=4\sqrt{2}$

$\theta=\arg(z)=tan^{-1}\left(\frac{4}{4}\right)=\frac{\pi}{4}$

Multiplying two complex numbers together:

$\left[2cis\left(\frac{5\pi}{6}\right)\right]\times\left[4\sqrt{2}cis\left(\frac{\pi}{4}\right)\right]=8\sqrt{2}cis\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)$

$=8\sqrt{2}cis\left(\frac{26\pi}{24}\right)=8\sqrt{2}cis\left(\frac{13\pi}{12}\right)$

3 Determine all roots, real and complex, of the equation $f(z)=z^3-4z^2+z+26$

Substitute different values of z until $f(z)=0$:

$f(0)=26\neq0$, $f(1)=24\neq0$, $f(-1)=20\neq0$, $f(2)=20\neq0\rightarrow$ these are not factors

$f(-2)=0$ hence $(z+2)$ is a factor

$\therefore z^3-4z^2+z+26=(z+2)(z^2+bz+c)$

Using polynomial long division (on page 2):

$propFrac\left(\frac{z^3-4z^2+z+26}{z+2}\right)=z^2-6z+13$

Find roots of $z^2-6z+13$ by quadratic formula:

$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{6\pm\sqrt{36-4(1)(13)}}{2(1)}$

$=\frac{6\pm\sqrt{-16}}{2}=\frac{6\pm\sqrt{16}\sqrt{-1}}{2}=\frac{6\pm4i}{2}=3\pm2i$

Hence roots are $z=-2,3+2i,3-2i$

4 Find all the complex numbers that satisfy the equation $|z|^2-iz=36+4i$

Let $z=x+yi$ and hence:

$|(x+yi)|^2-i(x+yi)=36+4i$

$\left(\sqrt{x^2+y^2}\right)^2-xi-yi^2=36+4i$

$x^2+y^2-xi+y-36-4i=0$

Equating real and imaginary parts:

$x^2+y^2+y-36=0$ and $-x-4=0$

Hence, $x=-4$ and $(-4)^2+y^2+y-36=0$

$16+y^2+y-36=0$

$y^2+y-20=0$ and $(y+5)(y-4)=0$

Giving $y=-5,4$ hence $z=-4-5i,-4+4i$

5 Let a and b be real numbers with $a\neq b$. If $z=x+yi$ such that $|z-a|^2-|z-b|^2=1$, prove that $x=\frac{a+b}{2}+\frac{1}{2(b-a)}$

$|(x+yi)-a|^2-|(x+yi)-b|^2=1$

$|(x-a)+yi|^2-|(x-b)+yi|^2=1$

$(x-a)^2+y^2-[(x-b)^2+y^2]=1$

$(x-a)^2-(x-b)^2=1$

$x^2-2ax+a^2-x^2+2bx-b^2=1$

$(2b-2a)x+a^2-b^2=1$

$x=\frac{1-a^2+b^2}{2b-2a}=\frac{a+b}{2}+\frac{1}{2(b-a)}$

DE MOIVRE'S THEOREM EXAMPLES

1 Find z^{10} given that $z=1-i$

$r=|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}$ and $\arg(z)=-\frac{\pi}{4}$

Hence, z in polar form is $z=\sqrt{2}cis\left(-\frac{\pi}{4}\right)$

Applying De Moivre's Theorem gives:

$z^{10}=\left(\sqrt{2}\right)^{10}cis\left(10\times-\frac{\pi}{4}\right)=2^5cis\left(-\frac{10\pi}{4}\right)$

$=32cis\left(-\frac{5\pi}{2}\right)=32cis\left(-\frac{5\pi}{2}+2\pi\right)$

$=32cis\left(-\frac{\pi}{2}\right)=32\left[cos\left(-\frac{\pi}{2}\right)+i\ sin\left(-\frac{\pi}{2}\right)\right]$

$=32[0+i(-1)]=-32i$

2 Use De Moivre's Theorem to find the smallest positive angle θ for which: $(cos\theta+i\ sin\theta)^{15}=-i$

$\cos(15\theta)+i\ sin(15\theta)=0-i$

Equating real and imaginary parts:

$0=\cos(15\theta)$ and $-1=sin(15\theta)$

Considering both conditions, $15\theta=\frac{3\pi}{2}$

Hence, $\theta=\frac{3\pi}{30}=\frac{\pi}{10}$ is the smallest positive angle

3 By expanding $(cos\theta+i\ sin\theta)^3$ show that $cos^3\theta=\frac{1}{4}cos3\theta+\frac{3}{4}cos\theta$

Step 1: expand the brackets of $(cos\theta+i\ sin\theta)^3$:

$(cos\theta+i\ sin\theta)^3=\cos^3\theta+3\cos^2\theta(i\ sin\theta)+3\cos(i\ sin\theta)^2+(i\ sin\theta)^3$

$=\cos^3\theta+3icos^2\theta sin\theta-3cos\theta\sin^2\theta-i\sin^3\theta$

Step 2: simplify $(cos\theta+i\ sin\theta)^3$ using De Moivre's Theorem:

$(cos\theta+i\ sin\theta)^3=cos3\theta+i\ sin3\theta$

Step 3: equating the real parts:

$\cos^3\theta-3cos\theta\sin^2\theta=cos3\theta$

$\cos^3\theta=cos3\theta+3cos\theta\,(1-\cos^2\theta)$

$\cos^3\theta=cos3\theta+3cos\theta-3\cos^3\theta$

$4\cos^3\theta=cos3\theta+3cos\theta$

$\cos^3\theta=\frac{1}{4}cos3\theta+\frac{3}{4}cos\theta$

DE MOIVRE'S THEOREM EXAMPLES

4 Find and graph all the complex fourth roots of -16 on an argand plane.

$r=|-16|=\sqrt{(-16)^2}=16$ and $\arg(-16)=\pi$

Hence, -16 in polar form is $z=16cis(\pi)$

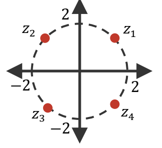
We need 4 roots hence $n=4$ and the roots are:

$z_1=16^{\frac{1}{4}}cis\left(\frac{\pi}{4}\right)=2cis\left(\frac{\pi}{4}\right)$

$z_2=16^{\frac{1}{4}}cis\left(\frac{\pi+(1\times2\pi)}{4}\right)=2cis\left(\frac{3\pi}{4}\right)$

$z_3=16^{\frac{1}{4}}cis\left(\frac{\pi+(2\times2\pi)}{4}\right)=2cis\left(\frac{5\pi}{4}\right)$

$z_4=16^{\frac{1}{4}}cis\left(\frac{\pi+(3\times2\pi)}{4}\right)=2cis\left(\frac{7\pi}{4}\right)$



Note that there are $n=4$ roots and that all roots are equally spaced out by an angle of $\frac{2\pi}{n}=\frac{2\pi}{4}=\frac{\pi}{2}$

5 One of the solutions of $z^3=a$, for some constant a, is $z=4\sqrt{3}-4i$. Determine all other solutions in Cartesian form.

$r^{1/3}=|4\sqrt{3}-4i|=\sqrt{\left(4\sqrt{3}\right)^2+(-4)^2}=8$ and

$\arg(4\sqrt{3}-4i)=\tan^{-1}\left(\frac{4}{-4\sqrt{3}}\right)=-\frac{\pi}{6}$

Hence, $4\sqrt{3}-4i$ in polar form is $z=8cis\left(-\frac{\pi}{6}\right)$

We need 3 roots hence $n=3$ and the roots are:

$z_1=8cis\left(-\frac{\pi}{6}\right)=4\sqrt{3}-4i$

$z_2=8cis\left(-\frac{\pi}{6}+\frac{2\pi}{3}\right)=8cis\left(\frac{3\pi}{6}\right)=8i$

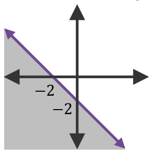
$z_3=8cis\left(-\frac{\pi}{6}+\frac{4\pi}{3}\right)=8cis\left(\frac{7\pi}{6}\right)=-4\sqrt{3}-4i$

TRANSFORMATIONS

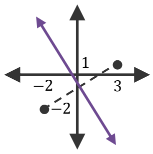
- Multiplying z by i rotates a complex number by 90º anti-clockwise.
- Multiplying z by i^n rotates a complex number by $\left(\frac{n\pi}{2}\right)$ anti-clockwise.
- Multiplying z by n increases the modulus of a complex number by scale factor n.
- Multiplying $Re(z)$ by -1 reflects a complex number in the y-axis.
- Multiplying $Im(z)$ by -1 reflects a complex number in the x-axis.

ARGAND (COMPLEX) PLANE

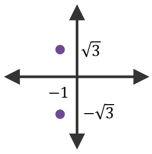
Draw the following on the complex plane:



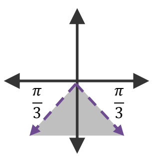
1 $\{z: |z+2+2i|\leq|z|\}$
$|(x+2)+i(y+2)|$
$\leq|x^2+y^2|$
$(x+2)^2+(y+2)^2$
$\leq x^2+y^2$
Simplifying this equation and making y the subject gives $y\leq-x-2$.



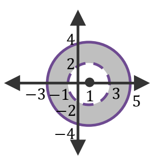
2 $\{z: |z+2+2i|=|z-3-i|\}$
$|z-(-2-2i)|$
$=|z-(3+i)|$
Place a point at (3,1) and $(-2,-2)$, find the halfway point between them and draw a perpendicular.



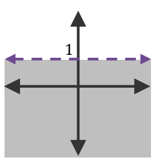
3 $\{z: z^2=-2z-4\}$
$z^2+2z+4=0$
Use quadratic equation to solve for z:
$z=-1\pm\sqrt{3}i$
$\therefore z=-1+\sqrt{3}i$ and
$z=-1-\sqrt{3}i$



4 $\left\{z: -\frac{\pi}{3}<arg(iz)<\frac{\pi}{3}\right\}$
$iz=i\times(x+yi)$
$xi+yi^2=xi-y$
$\therefore$ iz rotates a complex number by 90º anti-clockwise. Needs to be reversed in the answer.



5 $\{z: 2<|z-1|\leq4\}$
Draw a point at (1,0) and draw a doughnut with outer radius of 4 and inner radius of 2. Always take note of the inequality symbols used in the equation.

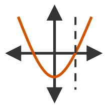


6 $\{z: z-\bar{z}<2i\}$
$(x+yi)-(x-yi)<2i$
$x+yi-x+yi<2i$
$2yi<2i$
$yi<i$
$y<1$
Take note of the inequality symbol used.

FUNCTIONS

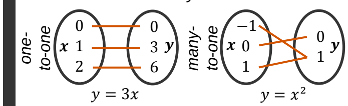
DEFINITION OF A FUNCTION

A function is one that:
- Passes the vertical line test



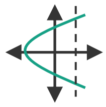
Vertical line cuts the curve once, so it passes the vertical line test. Therefore, this is a function.

- Is *one-to-one* or *many-to-one*



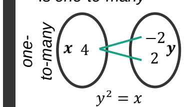
DEFINITION OF A NON - FUNCTION

A non-function (a.k.a. relation) is one that:
- Fails the vertical line test



Vertical line cuts the curve twice, so it fails the vertical line test. Therefore, this is not a function.

- Is *one-to-many*



$y^2=x$
$y=\pm\sqrt{x}$
$y=\sqrt{x}$ and $y=-\sqrt{x}$

COMPOSITE FUNCTIONS

Let $f(x)=ln(x^2+1)$ and $g(x)=2\sqrt{x}$, find:

1 $f\circ g(x)$

$=f(2\sqrt{x})=ln\left[\left(2\sqrt{x}\right)^2+1\right]=\ln(4x+1)$

2 Find $g(x)$ given $f\circ g(x)$ and $f(x)$

$f\circ g(x)=f(g(x))$

$\ln(4x+1)=ln(g(x)^2+1)$

Hence $g(x)^2=4x$ and $g(x)=\sqrt{4x}=2\sqrt{x}$

3 Find $f(x)$ given $f\circ g(x)$ and $g(x)$

Let $g(x)=2\sqrt{x}=u$

Solve $2\sqrt{x}=u$ for x: $x=\left(\frac{u}{2}\right)^2$

$f(g(x))=\ln(4x+1)=\ln\left[4\left(\frac{u}{2}\right)^2+1\right]$

$=\ln(u^2+1)\therefore f(u)=\ln(u^2+1)$

Change u to x: $f(x)=\ln(x^2+1)$

4 Let $f(x)=1+\sqrt{x-2}$ and $g(x)=\frac{1}{x-5}$, find the domain and range of $g\circ f(x)$

$g\circ f(x)=\frac{1}{1+\sqrt{x-2}-5}=\frac{1}{\sqrt{x-2}-4}$

Step 1: Find domain of inside function $f(x)$

Domain of $f(x)=\{x\in\mathbb{R}: x\geq2\}$

Step 2: Find domain of $g\circ f(x)$

Solve $\sqrt{x-2}-4\neq0$, $x-2\neq16$, $x\neq18$

Natural domain of $g\circ f(x)=\{x\in\mathbb{R}: x\neq18\}$

Step 3: The domain of $g\circ f(x)$ is the intersection of the two previous domains

Domain of $g\circ f(x)=\{x\in\mathbb{R}: x\geq2, x\neq18\}$

Step 4: To find the range of $g\circ f(x)$, analyse the critical points from the domain:
- For critical points that are $\leq,\geq$ substitute them directly into $g\circ f(x)$
- For critical points that are $\neq,<,>$ substitute a number that's ever so slightly lower and higher into $g\circ f(x)$
- Also substitute $\infty,-\infty$ into $g\circ f(x)$

$g\circ f(2)=-0.25$

$g\circ f(18.001)\rightarrow\infty$ and $g\circ f(17.999)\rightarrow-\infty$

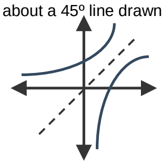
$g\circ f(\infty)\rightarrow0$ and $g\circ f(-\infty)=N/A$

Range of $g\circ f(x)=\{g\circ f(x)\in\mathbb{R}:$

$g\circ f(x)\leq-0.25, g\circ f(x)>0\}$

INVERSE FUNCTIONS

Inverse functions are diagonally symmetrical about a 45º line drawn through a set of axes.



Step 1: solve the function for x.
Step 2: swap all x's with y's, this new equation, is the inverse.

Inverse Function Rules

$f\circ f^{-1}(x)=f\left(f^{-1}(x)\right)=x$
$f^{-1}\circ f(x)=f^{-1}(f(x))=x$

1 Determine $f^{-1}(x)$ of $f(x)=ln(x+3)+1$

$f^{-1}(x)$ is the inverse of $f(x)$:

$f(x)=y=\ln(x+3)+1\rightarrow y-1=\ln(x+3)$

$e^{y-1}=x+3\rightarrow e^{y-1}-3=x\rightarrow y=e^{x-1}-3$

2 Prove that $f(x)=2x-3$ and $g(x)=0.5x+1.5$ are inverse functions.

$f(g(x))=2(0.5x+1.5)-3=x+3-3=x$

FUNCTIONS

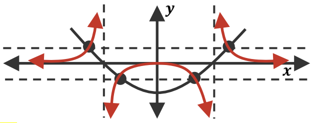
RECIPROCAL FUNCTIONS

Sketch $1/f(x)$ given $f(x)$

- Any x - intercepts on $f(x)$ are vertical asymptotes on $1/f(x)$
- Any intersections that $f(x)$ has with $y = 1$ or $y = -1$ are points on $1/f(x)$
- As $f(x)$ approaches ∞ or $-\infty$ it moves toward the x - axis on $1/f(x)$

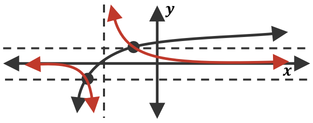
1 Sketch the function $y = \frac{1}{x^2-2}$

Let $f(x) = x^2 - 2$ and hence, $\frac{1}{f(x)} = \frac{1}{x^2-2}$



2 Sketch the function $y = \frac{1}{\ln(x+4)}$

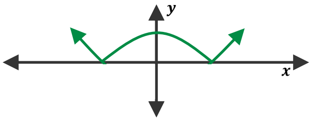
Let $f(x) = \ln(x+4)$ and hence, $\frac{1}{f(x)} = \frac{1}{\ln(x+4)}$



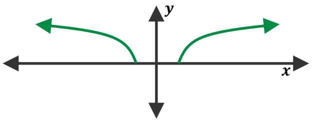
ABSOLUTE VALUE FUNCTIONS

- **Sketch $|f(x)|$:** Any points below the x - axis are reflected in the x - axis and any points above the x - axis aren't changed.
- **Sketch $f(|x|)$:** Reflects functions that cannot have negative x values (e.g. square root and logarithm functions) in the y – axis.

1 If $f(x) = x^2 - 3$, sketch $|f(x)|$



2 If $f(x) = \sqrt{x-2}$, sketch $f(|x|)$



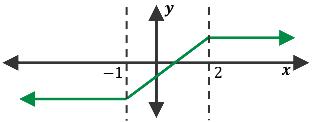
3 Sketch $y = |x+1| - |x-2|$

Solve each individual absolute value brackets for when it equals each individual absolute value brackets for when it equals 0: $|x+1| = 0, x = -1$ and $|x-2| = 0, x = 2$

Hence, $x = 1,2$ is the critical values.

Create a x/y table with each critical value above. Insert columns between each critical value and choose a random number between them. Solve the entire table for y:

x	-2	-1	0	2	3
y	-3	-3	-1	3	3



POLYNOMIAL FRACTION FUNCTIONS

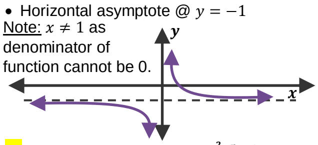
1 Sketch the function $y = \frac{-3+4x-x^2}{x^2-x}$

$$= \frac{-(x^2-4x+3)}{x(x-1)} = \frac{-(x-3)(x-1)}{x(x-1)}$$

$$= \frac{-(x-3)}{x} = \frac{3-x}{x} = \frac{3}{x} - 1$$

- Vertical asymptote @ $x = 0$
- Horizontal asymptote @ $y = -1$

Note: $x \neq 1$ as denominator of function cannot be 0.

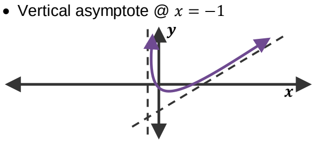


2 Sketch the function $y = \frac{x^2-5x+6}{x+1}$

Using polynomial long division (on the right):

$$propFrac\left(\frac{x^2-5x+6}{x+1}\right) = x - 6 + \frac{12}{x+1}$$

- Oblique asymptote @ $y = x - 6$
- Vertical asymptote @ $x = -1$



ABSOLUTE VALUE

Absolute Value Piecewise Function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

1 If $f(x) = x + 2$ and $g(x) = (x+1)^2 - 5$, solve $|f(x)| = |g(x)|$

$|g(x)| = |x^2 + 2x - 4| = |x+2| = |f(x)|$

Solving for when absolute value is positive:

$x^2 + 2x - 4 = x + 2 \to x^2 + x - 6 = 0$

$(x+3)(x-2) = 0 \to x = -3, 2$

Solving for when absolute value is negative:

$x^2 + 2x - 4 = -x - 2 \to x^2 + 3x - 2 = 0$

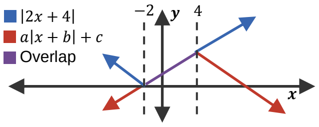
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{9+8}}{2}$$

$$= \frac{-3 \pm \sqrt{17}}{2} = 0.5616, -3.5616$$

$x = -3, 2, 0.5616, -3.5616$

2 If $|2x+4| = a|x+b| + c$, determine the values of the real constants a, b and c that over the domain $\{x \in \mathbb{R}: -2 \leq x \leq 4\}$

In order for two absolute functions to be equal over a given domain, the two functions cannot have the same concavity:



From the graph, we can find the signs for the values for a, b and c: a is negative (concave), b is negative (positive x - intercept) and c is positive (positive y – intercept). Hence, when $x = 4, y = 12 \to c = 12$. Also, $b = -4$ as there is a cusp at $x = 4$ and substituting $(-2, 0)$ into $y = a|x-4| + 12$ gives $a = -2$.

PARTIAL FRACTIONS

- Partial Fractions: ClassPad → Main → Action → Transformation → Expand

$expand(equation, x)$

- ClassPad output:

$$expand\left(\frac{3x+11}{x^2-x-6}, x\right) = \frac{4}{x-3} - \frac{1}{x+2}$$

1 Simplify $\frac{3x+11}{x^2-x-6}$

$$\frac{3x+11}{x^2-x-6} = \frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\frac{3x+11}{x^2-x-6} = \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

$3x + 11 = A(x+2) + B(x-3)$

$3x + 11 = Ax + 2A + Bx - 3B$

Hence, $3 = A + B$ and $11 = 2A - 3B$

Simultaneously solving on the ClassPad:

$A = 4, B = -1$

2 Simplify $\frac{x^2-29x+5}{(x-4)^2(x^2+3)}$

$$\frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$$

$x^2 - 29x + 5 = A(x-4)(x^2+3) + B(x^2+3) + (Cx+D)(x-4)^2$

$= (A+C)x^3 + (-4A + B - 8C + D)x^2 + (3A + 16C - 8D)x - 12A + 3B + 16D$

Equating co-efficients and solving:

$$\begin{array}{ll} x^3: & A + C = 0 \\ x^2: & -4A + B - 8C + D = 1 \\ x^1: & 3A + 16C - 8D = -29 \\ x^0: & -12A + 3B + 16D = 5 \end{array} \Rightarrow \begin{array}{l} A = 1 \\ B = -5 \\ C = -1 \\ D = 2 \end{array}$$

POLYNOMIAL LONG DIVISION

- Polynomial Long Division: ClassPad → Main → Action → Transformation → Fraction → propFrac

$propFrac(equation\ 1/equation\ 2)$

- ClassPad output:

$$propFrac\left(\frac{x^2-9x-10}{x+1}\right) = x - 10$$

1 Determine $\frac{3x^3-5x^2+10x-3}{3x+1}$

$$\begin{array}{rl} & x^2 - 2x + 4 \\ 3x+1\,) & 3x^3 - 5x^2 + 10x - 3 \\ - & 3x^3 + 1x^2 \\ & -6x^2 + 10x \\ - & -6x^2 - 2x \\ & +12x - 3 \\ - & +12x + 4 \\ & -7 \end{array}$$

$$= x^2 - 2x + 4 - \frac{7}{3x+1}$$

Step 1: divide the highest order polynomials and multiply this answer by the divisor.

Step 2: subtract the two equations

Step 3: repeat steps 1 and 2 until a single number remains.

3-D VECTORS

SYSTEMS OF LINEAR EQUATIONS

- Echelon Form: ClassPad → Main → Action → Matrix → Calculation → ref

$ref\left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}\right)$ This returns the matrix in echelon form.

- ClassPad output:

$$ref\left(\begin{bmatrix} 2 & 6 & 4 & 14 \\ 6 & 12 & 3 & 18 \\ 4 & 10 & 6 & 22 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 2 & 7 \\ 0 & 1 & 1.5 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- Reduced Echelon Form: ClassPad → Main → Action → Matrix → Calculation → rref

$rref\left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}\right)$ This returns the matrix in reduced echelon form (i.e. will give the answers for x, y and z).

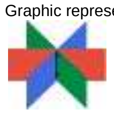
- ClassPad output:

$$rref\left(\begin{bmatrix} 2 & 6 & 4 & 14 \\ 6 & 12 & 3 & 18 \\ 4 & 10 & 6 & 22 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

SOLUTIONS OF LINEAR EQUATIONS

There are three types of solutions for a system of linear equations. To solve for these different solutions, the last row of matrix in echelon form must have the following forms:

- **Infinite Solutions:** more than 1 solution
  - Graphic representation:

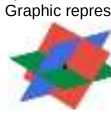


The three planes produce an intersection that is a line.

  - Last row of matrix in echelon has the form:

$[0 \quad 0 \quad 0 \quad | \quad 0]$

- **Unique Solution:** only 1 solution
  - Graphic representation:

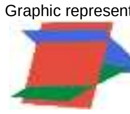


The three planes have a single point of intersection.

  - Last row of matrix in echelon has the form:

$[0 \quad 0 \quad A \quad | \quad B] \quad A, B \neq 0$

- **No Solutions:** 0 solutions
  - Graphic representation:



None of the three planes have a common intersection.

  - Last row of matrix in echelon has the form:

$[0 \quad 0 \quad 0 \quad | \quad B] \quad B \neq 0$

LINEAR EQUATIONS EXAMPLES

1 Reduce this matrix to echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 7+a & 5 \\ 2 & 3 & a^2+2 & a+10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 6+a & 2 \\ 0 & 1 & a^2 & a+4 \end{array}\right] \begin{array}{l} r_1 \\ r_2 - r_1 \\ r_3 - 2r_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 6+a & 2 \\ 0 & 0 & a^2-a-6 & a+2 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \\ r_3 - r_2 \end{array}$$

Note: ensure that row operations are written aside the matrix.

Using the matrix above, find a that gives:

2 No solutions

Last row in form of: $[0 \quad 0 \quad 0 \quad | \quad B] \; B \neq 0$

$\therefore a^2 - a - 6 = 0$ and $a + 2 \neq 0$

Solving to get $a = 3, -2$ and $a \neq -2$

$\therefore a = 3$ gives no solutions

3 Infinite solutions

Last row in form of: $[0 \quad 0 \quad 0 \quad | \quad 0]$

$\therefore a^2 - a - 6 = 0$ and $a + 2 = 0$

Solving to get $a = 3, -2$ and $a = -2$

$\therefore a = -2$ gives no solutions

4 A unique solution

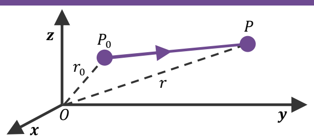
Last row in form of: $[0 \quad 0 \quad A \quad | \quad B] \; A, B \neq 0$

$\therefore a^2 - a - 6 \neq 0$ and $a + 2 \neq 0$

Solving to get $a \neq 3, -2$ and $a \neq -2$

$\therefore a \neq -2$ gives unique solution $(a \in \mathbb{R}: a \neq -2)$

DRAWING LINES



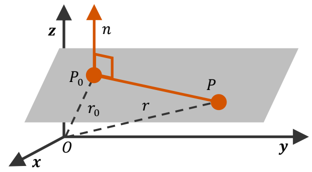
Parametric form of vector equation of a line

- $x = a + \lambda d$, $y = b + \lambda e$, $z = c + \lambda f$ where:
  - (a, b, c) is r_0 and (d, e, f) is $r - r_0$
  - λ determines the magnitude and direction

Cartesian equation of a line

- $\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}$ where:
  - (a, b, c) is r_0 and (d, e, f) is $r - r_0$

DRAWING PLANES



Vector Equation of a Plane

- $(r - r_0).n = 0$ where:
  - P and P_0 are points on the plane
  - n is normal (perpendicular) to the plane
  - This equation can be simplified to: $r.n - r_0.n = 0 \to r.n = r_0.n \to r.n = c$

Cartesian Equation of a Plane

- $Ax + By + Cz + D = 0$ where:
  - A, B, C and D are real-valued parameters
  - Vector (A, B, C) is normal (perpendicular) to the plane

VECTOR RULES

- Given $\tilde{x} = (a, b, c)$ and $\tilde{y} = (d, e, f)$:

General Vector Rules

$\overrightarrow{XY} = \tilde{y} - \tilde{x}$	$\lvert x \rvert = \sqrt{a^2 + b^2 + c^2}$
$\lvert\overrightarrow{XY}\rvert = \sqrt{(d-a)^2 + (e-b)^2 + (f-c)^2}$	

Unit Vector ($\hat{x}$)

- Returns vector with the same direction but with a magnitude of 1.

$\hat{x} = \frac{x}{\lvert x \rvert}$	$\lvert\hat{x}\rvert = 1$

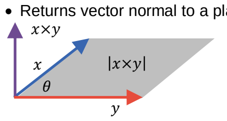
Dot Product ($x \, . \, y$)

- Dot product gives scalar result (a number).

$x \, . \, y = (a \times d) + (b \times e) + (c \times f)$	
$x \, . \, y = \lvert x \rvert \lvert y \rvert cos\theta$	$x . x = \lvert x \rvert^2$
x and y are perpendicular if $x \, . \, y = 0$	
$dotP([a, b, c], [d, e, f])$ ClassPad → Main → Action → Vector → dotP	

Cross Product ($x \times y$)

- Cross product gives vector result (a vector).
- Returns vector normal to a plane.



$|x \times y|$ is area of the parallelogram with sides x and y.

$x \times y = (bf - ce, cd - af, ae - bd)$
$x \times y = \hat{n}\lvert x \rvert \lvert y \rvert sin\theta$ Where $\hat{n}$ is the unit vector perpendicular to vectors x and y
$crossP([a, b, c], [d, e, f])$ ClassPad → Main → Action → Vector → crossP

VECTOR EXAMPLES

1 Vector equation of a line passing through two given points

Points A and B have co-ordinates $(2, 1, -3)$ and $(4, 5, -1)$ respectively.

$\overrightarrow{AB} = \tilde{b} - \tilde{a} = 2i + 4j + 2k$ and hence,

$r = (2i + j - 3k) + \lambda(2i + 4j + 2k)$

2 Test if a point is perpendicular to a line

Point to test is $A(1, 2, 1)$ and the equation of the line is $r = (i + 2j + 3k) + \lambda(4i + 2j - 8k)$

$(i + 2j + k).(4i + 2j - 8k) = 4 + 4 - 8 = 0$

Hence, the point is perpendicular to the line.

3 Intersection of two moving vectors

Find point of intersection between the lines $A = (-7i + 9j - 5k) + \lambda(5i - 4j + 2k)$ and $B = (-6i - 5j + 2k) + \mu(9i + 6j - 3k)$

Solve the i, j and k parts for λ and μ:

$-7 + 5\lambda = -6 + 9\mu$, $9 - 4\lambda = -5 + 6\mu$ and $-5 + 2\lambda = 2 - 3\mu$ and hence, $\lambda = 2$, $\mu = 1$ therefore point of intersection is $(3, 1, -1)$

4 Collision of two moving vectors

Find collision between moving vectors $A = (2i + 1j - 3k) + \lambda(7i + 10j - 3k)$ and $B = (5i + 28j - 6k) + \mu(6i + j - 2k)$ where velocity is measured in km/h.

Equating i – coefficients: $2 + 7\lambda = 5 + 6\mu$

Equating j – coefficients: $1 + 10\lambda = 28 + 1\mu$

Equating k – coefficients: $-3 - 3\lambda = -6 - 2\mu$

Solving the first two equations (i and j coefficients) for λ and μ: $\lambda = 3$ and $\mu = 3$

Substitute into third equation (k coefficient): $-3 - 3(3) = -6 - 2(3) \to 6 = 6$ which is consistent so a collision occurs as times λ and μ are the same (@ $t = 3$). Finding collision point, substitute $t = 3$ back into A or B:

$A = (2i + 1j - 3k) + 3(7i + 10j - 3k)$

$= (23i + 31j - 12k)$

VECTOR EXAMPLES

5 Intersection of two moving vectors
Find intersection between moving vectors $A = (-7i + 9j - 5k) + \lambda(5i - 4j + 2k)$ and $B = (-6i - 5j + 2k) + \mu(9i + 6j - 3k)$

Equating i – coefficients: $-7 + 5\lambda = -6 + 9\mu$
Equating j – coefficients: $9 - 4\lambda = -5 + 6\mu$
Equating k – coefficients: $-5 + 2\lambda = 2 - 3\mu$
Solving the first two equations (i and j coefficients) for λ and μ: $\lambda = 2$ and $\mu = 1$
λ and μ are different hence intersection at $A = (-7i + 9j - 5k) + 2(5i - 4j + 2k)$
$= (3i + j - k)$

6 Shortest distance between two moving vectors Find shortest distance between the two moving vectors where velocity is measured in km/h.
$A = (2i + j - 3k) + \lambda(7i + 10j - 3k)$ and $B = (-5i + 20j + k) + \mu(-3i - j + 7k)$

$\vec{d} = \overrightarrow{BA} + ({}_AV_B)t$ and $\vec{d}\,.\,{}_AV_B = 0$ where:

- $\vec{d}$: shortest displacement between A and B
- $\overrightarrow{BA} = \tilde{a} - \tilde{b}$: vector between A and B
- ${}_AV_B = V_A - V_B$: relative velocity of B to A

$\overrightarrow{BA} = \begin{bmatrix}7\\-19\\-4\end{bmatrix}$ and ${}_AV_B = \begin{bmatrix}7\\10\\-3\end{bmatrix} - \begin{bmatrix}-3\\-1\\7\end{bmatrix} = \begin{bmatrix}10\\11\\-10\end{bmatrix}$

$\vec{d} = \overrightarrow{BA} + ({}_AV_B)t = \begin{bmatrix}7\\-19\\-4\end{bmatrix} + t\begin{bmatrix}10\\11\\-10\end{bmatrix}$

Using ClassPad to find time, $\vec{d}\,.\,{}_AV_B$

$= dotP\left(\begin{bmatrix}7\\-19\\-4\end{bmatrix} + t\begin{bmatrix}10\\11\\-10\end{bmatrix}, \begin{bmatrix}10\\11\\-10\end{bmatrix}\right) = 0.308\ hr$

Using ClassPad to find distance,

$= \left|\begin{bmatrix}7\\-19\\-4\end{bmatrix} + 0.308\begin{bmatrix}10\\11\\-10\end{bmatrix}\right| = 19.89km$

7 Vector equation of a plane
A plane contains the point $(5, -7, 2)$ and has a normal parallel to $(3, 0, -1)$

$\begin{bmatrix}x-5\\y+7\\z-2\end{bmatrix}.\begin{bmatrix}3\\0\\-1\end{bmatrix} = 0$ hence, $\begin{bmatrix}x\\y\\z\end{bmatrix}.\begin{bmatrix}3\\0\\-1\end{bmatrix} = 13$

8 Locating where a line intersects with a plane A plane contains the point $(5, -7, 2)$ and has a normal parallel to $(3, 0, -1)$, where does it intersect with the line $A = (-10i + 4j - 9k) + \lambda(2i + j - 6k)$

$\begin{bmatrix}-10+2\lambda\\4+\lambda\\-9-6\lambda\end{bmatrix}.\begin{bmatrix}3\\0\\-1\end{bmatrix} = 13$ Solving on ClassPad

$\lambda = 17/6$ and substituting into $A = \begin{bmatrix}-26/6\\41/6\\-26\end{bmatrix}$

9 Equation of a plane using three non-collinear points Find the equation of a plane that passes through the points $A(1, 1, 1)$, $B(-1, 1, 0)$ and $C(2, 0, 3)$

$\overrightarrow{AB} = (-2, 0, -1)$ and $\overrightarrow{AC} = (1, -1, 2)$
$\overrightarrow{AB} \times \overrightarrow{AC} = (-1, 3, 2)$ and hence equation of the plane is $-x + 3y + 2z + D = 0$. Sub any point to find D: $-(2) + 3(0) + 2(3) + D = 0$
$D = -4$ hence $-x + 3y + 2z - 4 = 0$

10 Cartesian equation of a sphere
Find the radius and co-ordinates of the centre of the sphere with the equation
$x^2 + y^2 + z^2 + 2x + 4y - 6z - 50 = 0$

$x^2 + y^2 + z^2 + 2x + 4y - 6z = 50$
$LHS = (x+1)^2 + (y+2)^2 + (z-3)^2$
$RHS = 50 + 1 + 4 + 9 = 64 = 8^2$
Hence, centre at $(-1, -2, 3)$ and radius of 8.

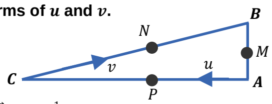
11 Cartesian equation of a hyperbola
Find the cartesian equation of the hyperbola with the vector equation
$A = [3\tan(t)]i + [4\sec(t)]j$

$\frac{x}{3} = \tan(t)$ and $\frac{y}{4} = \sec(t)$

$1 + \tan^2\theta = \sec^2\theta \to 1 + \left(\frac{x}{3}\right)^2 = \left(\frac{y}{4}\right)^2$

$1 + \frac{x^2}{9} = \frac{y^2}{16} \to \frac{y^2}{16} - \frac{x^2}{9} = 1$

12 Vectors in practice
Triangle ABC is below with the midpoints of each side M, N and P shown. Let $\overrightarrow{AC} = u$ and $\overrightarrow{CB} = v$. Express $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP}$ in terms of u and v.



$\overrightarrow{AN} = u + \frac{1}{2}v$

$\overrightarrow{CM} = -u + \frac{1}{2}(u + v) = \frac{1}{2}v - \frac{1}{2}u$

$\overrightarrow{BP} = -\frac{1}{2}u - v$

$\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP} = u + \frac{1}{2}v + \frac{1}{2}v - \frac{1}{2}u + -\frac{1}{2}u - v = u - u + v - v = 0$

TRIGONOMETRY IDENTITIES

Reciprocal Identities

$\sin(x) = \frac{1}{cosec(x)}$	$\cos(x) = \frac{1}{sec(x)}$	$\tan(x) = \frac{1}{cot(x)}$
$cosec(x) = \frac{1}{sin(x)}$	$\sec(x) = \frac{1}{cos(x)}$	$\cot(x) = \frac{1}{tan(x)}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$

Quotient Identities

$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\cot(x) = \frac{\cos(x)}{\sin(x)}$

Co-Function Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$	$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$

Parity Identities (Even and Odd)

$\sin(-x) = -sin(x)$	$\cos(-x) = \cos(x)$
$\tan(-x) = -\tan(x)$	$\sec(-x) = \sec(x)$

Sum and Difference

$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$

Double Angle

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
$\sin(2x) = 2\sin(x)\cos(x)$
$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$

Power Reducing

$\sin^2(x) = \frac{1 - \cos(2x)}{2}$	$\cos^2(x) = \frac{1 + \cos(2x)}{2}$

Limits of Sine and Cosine

$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$	$\lim_{x\to 0}\frac{1 - cos(x)}{x} = 0$

DIFFERENTIATION RULES

Product, Quotient and Chain Rules

$y = uv \to \frac{dy}{dx} = u'v + uv'$
$y = \frac{u}{v} \to \frac{dy}{dx} = \frac{u'v - uv'}{v^2}$
$y = [f(x)]^n \to \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

Common Derivatives

$y = ax^n \to \frac{dy}{dx} = n \times ax^{n-1}$
$y = e^{f(x)} \to \frac{dy}{dx} = f'(x) \times e^{f(x)}$
$y = \frac{1}{x} = x^{-1} \to \frac{dy}{dx} = \frac{-1}{x^2} = -x^{-2}$
$y = \pm sin(x) \to \frac{dy}{dx} = \pm cos(x)$
$y = \pm cos(x) \to \frac{dy}{dx} = \mp sin(x)$
$y = \pm tan(x) \to \frac{dy}{dx} = \pm\sec^2(x) = \frac{\pm 1}{\cos^2(x)}$
$y = \ln[f(x)] \to \frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^x \to \frac{dy}{dx} = \ln(a) \times a^x$

INTEGRATION RULES

Integral Rules

$\int_a^b f(x) = -\int_b^a f(x)$
$\int ax^n\, dx = a\int x^n\, dx$

Fundamental Theorem of Calculus

$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$
$\int_a^b f'(x)dx = f(b) - f(a)$

Integration by Parts

$\int uv'\, dx = uv - \int u'v\, dx$

Area Between Curves

$\int_a^b \frac{upper}{curve}dx - \int_a^b \frac{lower}{curve}dx$

INTEGRATION RULES

Common Integrals

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$\int f'(x) \times [f(x)]^n\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$
$\int e^{f(x)}\, dx = \frac{e^{f(x)}}{f'(x)} + c$
$\int \frac{f'(x)}{f(x)}\, dx = \ln\lvert f(x)\rvert + c$
$\int sin(x)\, dx = -cos(x) + c$
$\int cos(x)\, dx = sin(x) + c$
$\int sec^2(x)\, dx = tan(x) + c$

IMPLICIT DIFFERENTIATION

1 The point (a, b) lies on the curves $x^2 - y^2 = 5$ and $xy = 6$. Prove that the tangents to these curves at (a, b) are perpendicular.

Differentiating $x^2 - y^2$ with respect to x:
$x^2 - y^2 = 5 \to 2x - 2y\frac{dy}{dx} = 0 \to \frac{dy}{dx} = \frac{x}{y}$
At point (a, b) the slope is $m_1 = \frac{x}{y}$
Differentiating $xy = 6$ with respect to x:
$xy = 6 \to y + x\frac{dy}{dx} = 0 \to \frac{dy}{dx} = -\frac{y}{x}$
At point (a, b) the slope is $m_2 = -\frac{y}{x}$
Lines are perpendicular if $m_1 \times m_2 = -1$
$m_1 \times m_2 = \frac{x}{y} \times -\frac{y}{x} = -1$

2 Find the gradient at the point $(2, -1)$ on the curve $x + x^2y^3 = -2$

Differentiating with respect to x:
$1 + 2xy^3 + x^2 3y^2\frac{dy}{dx} = 0 \to \frac{dy}{dx} = \frac{-1 - 2xy^3}{x^2 3y^2}$
$\left.\frac{dy}{dx}\right|_{x=2, y=-1} = \frac{-1 - 2\times 2\times(-1)^3}{2^2 \times 3 \times (-1)^2} = \frac{1}{4}$

3 Determine the derivative of $\sqrt{x} + \sqrt{y} = 1$

Differentiating with respect to x:
$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \to \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

4 Find the co-ordinates of the points where the tangent to the curve $x^2 + 2xy + 3y^2 = 18$ is horizontal.

Differentiating with respect to x:
$2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$
$\frac{dy}{dx}(2x + 6y) = -2x - 2y = -\left(\frac{x + y}{x + 3y}\right)$
Solve for when $\frac{dy}{dx} = 0$ hence $x = -y$
Substitute into original: $y^2 - 2y^2 + 3y^2 = 18$
$y^2 = 9$ and hence, $y = \pm 3, x = \pm 3$

$\frac{80}{(20 + x)(20 - x)} = \frac{A(20 - x) + B(20 + x)}{40 - x^2}$
$80 = 20A - Ax + 20B + Bx$
Hence, $80 = 20A + 20B$ and $0 = B - A$
$A = 2, B = -2$ hence, integral is $\int \frac{2}{20+x} - \frac{2}{20-x}dx$
$= 2ln(\lvert 20 + x\rvert) - 2ln(\lvert 20 - x\rvert) + c$

DIFFERENTIAL EQUATIONS

1 A solution of a differential equation is $y = Ae^{-2t} + Be^{-t}$. When $t = 0$, it given that $y = 0$ and $\frac{dy}{dt} = 1$. Find the values of A and B.

$y = Ae^{-2t} + Be^{-t} \to \frac{dy}{dx} = -2Ae^{-2t} - Be^{-t}$
Using that $y = 0$ when $t = 0$: $0 = A + B$
Using that $\frac{dy}{dx} = 1$ when $t = 0$: $-1 = -2A + B$
Solving for A and B: $A = -1$ and $B = 1$
Hence, $y = -e^{-2t} + e^{-t}$

2 Determine the equation of the graph from the following conditions:

- **Gradient of the tangent at all points is given by $-\frac{x}{3y}$**
- **The graph passes through $(3, 1)$**

$\frac{dy}{dx} = -\frac{x}{3y} \to \int 3ydy = \int -xdx$
$\frac{3y^2}{2} = -\frac{x^2}{2} + C \to 3y^2 = -x^2 + C$
Applying initial condition (3,1):
$3(1)^2 = -(3)^2 + C \to 3 = -9 + C \to C = 12$
Hence, $2y^2 = 3x^2 + 12$

3 Determine the general solution for $y' = 6y^2x$ given that $x = 1, y = \frac{1}{25}$

$\frac{dy}{dx} = 6y^2x \to \int\frac{dy}{y^2} = \int 6xdx \to -\frac{1}{y} = 3x^2 + c$
Applying initial condition (1/25,1):
$-25 = 3 + c \to c = -28$ hence, $-\frac{1}{y} = 3x^2 - 28$

LOGISTIC EQUATION

Logistic Equation Differential Equation

- Used in biology, mathematics, economics, chemistry, probability and statistics

Form	$\frac{dP}{dt} = aP - bP^2$
Solution	$P = \frac{a}{b + ke^{-at}}$

1 Show that if $P = \frac{a}{b + ke^{-at}}$, then the derivative is in the form $\frac{dP}{dt} = aP - bP^2$

From these two equations, deduce that:
$ke^{-at} = \frac{a}{P} - b$

$\frac{dP}{dt} = a\left(\frac{a}{b + ke^{-at}}\right) - b\left(\frac{a}{b + ke^{-at}}\right)^2$
$= \frac{a^2}{b + ke^{-at}} - \frac{a^2b}{(b + ke^{-at})^2}$
$= \frac{a^2}{b + ke^{-at}}\left[\frac{1}{1} - \frac{b}{b + ke^{-at}}\right]$
$= \frac{a^2}{b + ke^{-at}}\left[\frac{b + ke^{-at} - b}{b + ke^{-at}}\right]$

$= \frac{a^2ke^{-at}}{(b + ke^{-at})^2} = \frac{a^2\left(\frac{a}{P} - b\right)}{\left(\frac{a}{P}\right)^2} = aP - bP^2$

2 If $\frac{dP}{dt} = 0.2P - 0.002P^2$, determine P as a function of t from question 1 above given that when $t = 0, P = 5$.

$\frac{0.2}{0.002 + ke^0} = 5 \to k = 0.038$
$\therefore P = \frac{0.2}{0.002 + 0.038e^{-0.2t}}$

VECTOR AND MOTION CALCULUS

Displacement, Velocity and Acceleration

Displacement	$r(t)$
Velocity	$v(t) = r'(t)$
Acceleration	$a(t) = v'(t) = r''(t)$

1 A particle is moving in m/s along a straight line and the acceleration of the particle is modelled by $a(t) = 2 - e^{\frac{-x}{2}}$. When $v = 4$, $x = 0$. Find v^2 in terms of x.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a(t) = 2 - e^{\frac{-x}{2}}$
$\frac{1}{2}v^2 = \int 2 - e^{\frac{-x}{2}}dx = 2x + 2e^{\frac{-x}{2}} + c$
When $v = 4$, $x = 0$ hence,
$\frac{1}{2}(16) = 0 + 2 + c$, $c = 6$
$\therefore \frac{1}{2}v^2 = 2x + 2e^{\frac{-x}{2}} + 6v^2 = 4x + 4e^{\frac{-x}{2}} + 12$

The position vector of a particle is initially at $r = -j$ cm and is moving horizontally with velocity in cm/s according to the equation $v = (3cost)i + (sint)j$

2 What is the initial acceleration?

$a(t) = v'(t) = (-3sint)i + (cost)j$

3 Find the displacement function.

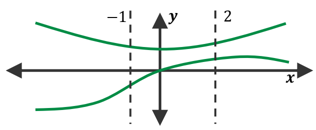
$r(t) = \int v(t)dt = (3sint)i - (cost)j + c$
As initially $r = -j$, $c = 0$ hence:
$r(t) = (3sint)i - (cost)j$

3 Determine the cartesian equation of the path of the particle.

$sint = \frac{x}{3}$ and $cost = -y$
$\sin^2 t + \cos^2 t = \left(\frac{x}{3}\right)^2 + (-y)^2 = 1$
$\frac{x^2}{9} + y^2 = 1$

AREA BETWEEN CURVES

1 Determine the area between the two curves $f(x) = x^2 + 2$ and $g(x) = sin(x)$ with the condition $-1 \le x \le 2$



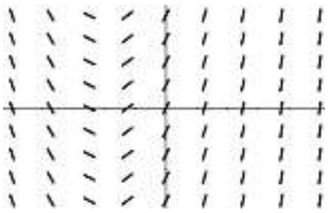
Upper curve is $f(x)$ and the lower curve is $g(x)$ with the bounds $x = -1$ and $x = 2$.

Hence, $A = \int_{-1}^{2} f(x) - g(x)dx$
$= \int_{-1}^{2} (x^2 + 2) - (\sin(x))dx$
$= \left[\frac{1}{3}x^3 + 2x + \cos(x)\right]_{-1}^{2}$
$= \left[\left(\frac{1}{3}(2)^3 + 2(2) + \cos(2)\right) - \left(\frac{1}{3}(-1)^3 + 2(-1) + \cos(-1)\right)\right] = 8.04$

CALCULUS

SLOPE (GRADIENT FIELDS)

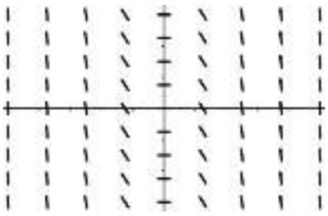
1 Determine a general differential equation for the following slope field and explain your reasoning.



$$\frac{dy}{dx} = ax + b$$

- Quadratic equation formed by isoclines.
- Convex nature, hence a is positive.
- x-intercept on the negative x-axis, hence b is positive.

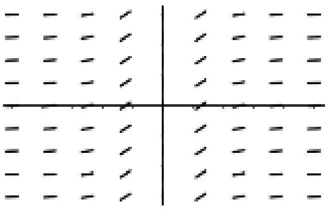
2 Determine a general differential equation for the following slope field and explain your reasoning.



$$\frac{dy}{dx} = -ax^2 - b$$

- Isoclines are all have negative gradient, hence cubic function.
- Point of inflection is on the y-axis.
- Consistent negative isoclines indicate negative gradient.

3 Determine a general differential equation for the following slope field and explain your reasoning.



$$\frac{dy}{dx} = \frac{a}{x^2} + b$$

- Hyperbolic function formed by isoclines.
- Gradient is ∞ at $x = 0$, hence vertical asymptote at $x = 0$
- Power of x must be even as gradient of positive x-values is positive as well as negative x-values.

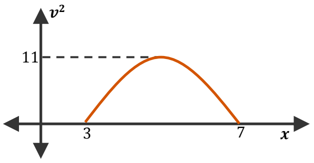
SIMPLE HARMONIC MOTION

Simple Harmonic Motion Rules

- A: amplitude of the motion
- α or β: angles of phase
- v: velocity and x: displacement

$\frac{d^2x}{dt^2} = -k^2x$	
$x = Asin(kt + \alpha)$	$x = Acos(kt + \beta)$
$v^2 = k^2(A^2 - x^2)$	

1 A particle is moving in m/s along the x-axis in simple harmonic motion. The parabola below shows v^2 as a function of x.



Determine the values of a, c and n in the equation $v^2 = n^2(a^2 - (x - c)^2)$.

$c = 5$ as the particle oscillated about $x = 5$

$a = 2$ as the amplitude is $5 - 3 = 2$ or $7 - 5 = 2$

Hence, $11 = n^2(4 - (x - 5)^2)$

As $v^2 = 11$ when $x = 5$, $11 = 4n^2 \to n = \frac{\sqrt{11}}{2}$

INCREMENTAL FORMULA

Incremental Formula (small change)

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

1 A differential equation has a point at $(5, 6)$ and $\frac{dy}{dx} = xy - x^2$. Determine an estimate for y when $x = 5.2$.

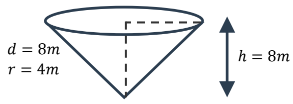
Using Euler's method with $\delta x = 0.1$

x	y	$\frac{dy}{dx}$	$\delta y \approx \frac{dy}{dx} \times \delta x$
5	6	5	0.5
5.1	6.5	7.14	0.714
5.2	7.214		

Estimate is $y = 7.214$

RELATED RATES

1 An inverted cone $8m$ tall has an upper diameter of $8m$ and is filling with water at a rate of $2m^3/min$. At what rate is the water level rising in the container when the depth of water is exactly $3.5m$?



From the question, substitute $r = \frac{h}{2}$ into volume

$\therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3 \to \frac{dV}{dh} = \frac{1}{4}\pi h^2$

To find $\frac{dh}{dt}$ when $h = 3.5$:

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{4}{\pi h^2} \times 2 = \frac{4}{\pi(3.5)^2} \times 2$

$= \frac{8}{\pi(3.5)^2} = \frac{32}{49\pi} m/min$

VOLUMES OF REVOLUTION

Revolution about the x-axis

- a and b: are bounds on the x-axis

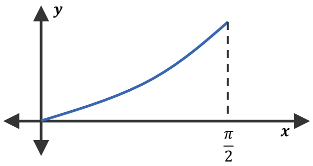
$$V = \pi \int_a^b y^2 dx$$

Revolution about the y-axis

- a and b: are bounds on the y-axis

$$V = \pi \int_a^b x^2 dy$$

1 Determine the region bounded by the line $x = \frac{\pi}{2}$ and $y = 3tan\left(\frac{x}{3}\right)$ rotated around the x-axis.



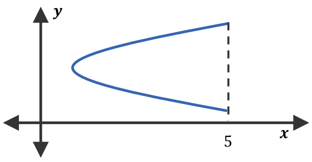
$V = \pi \int_a^b y^2 dx = \pi \int_0^{\pi/2} \left(3tan\left(\frac{x}{3}\right)\right)^2 dx$

$\left(3tan\left(\frac{x}{3}\right)\right)^2 = 9\tan^2\left(\frac{x}{3}\right) = 9\sec^2\left(\frac{x}{3}\right) + 9$

$= \pi \int_0^{\frac{\pi}{2}} 9\sec^2\left(\frac{x}{3}\right) - 9\, dx$

$= \pi \left[27tan\left(\frac{x}{3}\right) - 9x\right]_0^{\frac{\pi}{2}} = \frac{-9\pi}{2} + 9\sqrt{3}\pi$

2 Determine the volume of the region in between the functions $x = y^2 - 6y + 10$ and $x = 5$ rotated around the y-axis.



Determine the points of intersection:

$5 = y^2 - 6y + 10 \to 0 = y^2 - 6y + 5$

$0 = (y - 5)(y - 1) \to y = 1, 5$

Hence, points of intersection are $(5,1)$ and $(5,5)$

Inner radius $= y^2 - 6y + 10$

Outer radius $= 5$

Revolution around y-axis $= \pi \int_a^b x^2 dy$

Hence, this question can be treated as an area between two curves question with respect to the y-axis.

$\therefore x^2 = [(outer\ radius)^2 - (inner\ radius)^2]$

$= [(5)^2 - (y^2 - 6y + 10)^2]$

$= [-75 + 120y - 56y^2 + 12y^3 - y^4]$

Finding volume:

$V = \pi \int_1^5 -75 + 120y - 56y^2 + 12y^3 - y^4 dy$

$= \pi \left[-75y + 60y^2 - \frac{56}{3}y^3 + 3y^4 - \frac{1}{5}y^5\right]_1^5$

$= \frac{1088}{15}\pi = 227.87\ units^2$

STATISTICAL INFERENCE

RANDOM SAMPLES

Population Notation

- μ: population mean
- σ: population standard deviation
- σ^2: variance

Sample Notation

- $\bar{X}$: sample mean
- n: sample size
- If $n \geq 30$, regardless of the prior distribution, the sample data will become normally distributed with parameters:
  - Mean: $\bar{X}$
  - Standard Deviation: $\frac{\sigma}{\sqrt{n}}$

Z-Score $Z \sim N(0, 1)$

$$Z = \frac{X - \mu}{\sigma}$$

Sample Size

- d: value of the difference from the mean.

$$n = \left(\frac{z \times \sigma}{d}\right)^2$$

CONFIDENCE INTERVALS

Confidence Intervals

- z: z-score for a given confidence interval

$$\bar{X} - z\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z\frac{\sigma}{\sqrt{n}}$$

Common Confidence Intervals (z-scores)

99% CI	2.58
95% CI	1.96
90% CI	1.645

- Custom Confidence Interval: ClassPad → Main → Action → Distribution → Inverse → invNormCDf

$z_c = -1\times$ invNormCDf("C", c, 1, 0)

Where c is the CI% as a decimal.

STATISTICAL INFERENCE EXAMPLES

1 Determine a 95% confidence interval of a sample of 25 results with mean of 20 and variance of 4.

$20 - 1.96\left(\frac{2}{\sqrt{25}}\right) \leq \mu \leq 20 + 1.96\left(\frac{2}{\sqrt{25}}\right)$

Hence, the 95% CI is $[19.216, 20.784]$

2 What size sample is needed to ensure that sample mean is within 1.5 of the population mean with 99% confidence, given the standard deviation is 13.

$n = \left(\frac{z\times\sigma}{d}\right)^2 = \left(\frac{2.58\times 13}{1.5}\right)^2 = 499.96 \approx 500$

3 How large of a sample is needed to be 95% confident that the sample mean is within 10 of the population mean, given the standard deviation is 15.

$10 = 1.96\left(\frac{15}{\sqrt{n}}\right) \to n = 8.6436 \approx 9$

4 45 samples of mean 94 and standard deviation 12 was taken. Determine the parameters of the normal distribution.

$X \sim N\left(94, \left(\frac{12}{\sqrt{45}}\right)^2\right)$

YOUR NOTES AND EXAMPLES